

Temperatures of renormalizable quantum field theories in curved spacetime

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We compute the instantaneous temperature registered by an Unruh-DeWitt detector coupled to a Hadamard renormalizable massless quantum field in a generic state, which is moving along an accelerated trajectory in curved spacetime. The general expression for the temperature depends on the 4-acceleration, Raychaudhuri scalar, and renormalized field polarization. We can further find a novel constraint on the renormalized quantum field polarization in relativistic systems in global thermal equilibrium.

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I. INTRODUCTION

One of the key insights into quantum gravity provided by quantum field theory in curved spacetime is its prediction that varying geometry can give rise to thermal particle production [1–3]. This particle production provides a bridge between thermodynamics, quantum theory, and general relativity through the associated temperature. To compute this temperature one can use, e.g. an Unruh-DeWitt detector [3], which requires knowing the Wightman function, i.e. two point function, for the specific field, background geometry, and acceleration in question. Here we use the Hadamard form [4] of the two point function to compute the temperature, for a generic Hadamard state. This has the advantage of encompassing all types of fields and geometries which are renormalizable using the Hadamard prescription of subtracting off the singularity structure of the Hadamard form from the two point function. Using an Unruh-DeWitt detector enables the use of the singularity structure to determine the temperature. To focus on this singularity structure we employ a covariant expansion [5] of the Hadamard form evaluated in the quasilocal limit [6]. This enables us to include contributions from the background curvature, the quantum state, as well as the proper acceleration of the detector. The resultant expression for the temperature is a generalization of the Page approximation [7] and relates the local temperature to the renormalized vacuum polarization, acceleration, and Raychaudhuri scalar. Furthermore, we can put lower limit on the temperature seen by a local observer, in terms of its trajectory and the local spacetime curvature, independent of the quantum state.

II. THE COVARIANT AND QUASILOCAL EXPANSION OF THE HADAMARD FORM

Numerous observables in quantum field theory are computed using the two point function and its variants. An observable of particular importance is the expectation value of the quantum energy momentum tensor which is used as the source of the semiclassical Einstein equation. Although the computation of the energy momentum tensor is formally infinite, the Hadamard renormalization prescription, i.e. point splitting, renders the resultant expression finite. This is accomplished by subtracting the singularity structure of the Hadamard form from the two point function used in computing the energy momentum tensor. Another observable of interest is the temperature registered by an Unruh-DeWitt detector. Interestingly enough, it is precisely the singularity structure of the two point function, or more specifically the pole and residue structure, which is needed to compute the temperature. By using the Hadamard form, we are able to use the singularity structure of all renormalizable quantum field theories to compute the temperature. The Hadamard form for the symmetrized Wightman function along a trajectory $x(\tau)$ is given by [8]

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$$G^+[x(\tau), x(\tau')] \equiv \frac{1}{2} \langle \{ \phi[x(\tau)], \phi[x(\tau')] \} \rangle = \frac{1}{8\pi^2} \left[\frac{\Delta^{1/2}}{\sigma} + v \ln(\sigma) + w \right]. \quad (1)$$

Here, the biscalars v and w respectively characterize the quantum field's potential, quantum state, and have the following covariant expansions; $v = \sum_n v_n(x, x') \sigma^n$ and $w = \sum_n w_n(x, x') \sigma^n$. This polynomial expansion is expressed in terms of the covariant derivative of Synge's world function $\nabla_\alpha \sigma(x, x') \equiv \sigma_\alpha$, where 2σ is the square of geodesic distance between x and x' . Expressing both σ and σ_α utilizing the quasilocal expansion [6] we have

$$\begin{aligned} \sigma &= -\frac{1}{2}s^2 - \frac{1}{24}A^2s^4 + \mathcal{O}(s^5) \\ \sigma^\mu &= -su^\mu - \frac{s^2}{2}a^\mu + \mathcal{O}(s^3). \end{aligned} \quad (2)$$

Here we have defined the acceleration $A^2 = g_{\mu\nu} \frac{Du^\mu}{d\tau} \frac{Du^\nu}{d\tau} = g_{\mu\nu} a^\mu a^\nu$ and made use of the identity $2\sigma = \sigma_\mu \sigma^\mu$ [9]. The parameter $s = \tau' - \tau$ characterizes the proper time of the detector. The overall minus sign indicates that the detector moves along a timelike trajectory normalized via $u^\alpha u_\alpha = -1$. The covariant expansion of the Van Vleck Moretti determinant are given by [5]

$$\Delta^{1/2} = 1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta + \mathcal{O}(\sigma^{3/2}) \quad (3)$$

Note that if we use the elementary averaging procedure [10] in the Van Vleck Moretti determinant to exchange the Ricci tensor for the Ricci scalar via $R_{\alpha\beta} \sigma^\alpha \sigma^\beta \rightarrow R_{\alpha\beta} \frac{1}{2} g^{\alpha\beta} \sigma = \frac{1}{2} R \sigma$, then the expansion is given by $\Delta^{1/2} = 1 + \frac{1}{24} R \sigma$.

Finally, if we rewrite the Hadamard form with Synge's [9] world function as a common denominator and expanding to first order in σ we have

$$G^+[x(\tau'), x(\tau)] = \frac{1}{8\pi^2} \left[\frac{1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta + v_0 \sigma \ln(\sigma) + w_0 \sigma}{\sigma} \right] \quad (4)$$

Note that we now have the zeroth order, i.e. coincident limit, terms for the potential and state dependent terms v_0 and w_0 . The potential is given by $v_0 = m^2 + (\xi - \frac{1}{6})R$ while the state dependence can be written in terms of the renormalized vacuum polarization [11] via $w_0 = 8\pi^2 \langle \phi^2 \rangle_{ren} - v_0 \ln(\mu)$. Here we introduce a mass scale μ to render the logarithm term in the Hadamard form dimensionless. Finally we have

$$G^+[x(\tau'), x(\tau)] = \frac{1}{8\pi^2} \left[\frac{1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta + v_0 \sigma \ln(\tilde{\sigma}) + 8\pi^2 \sigma \langle \phi^2 \rangle_{ren}}{\sigma} \right] \quad (5)$$

Here we have defined the dimensionless world function via $\tilde{\sigma} = \sigma/\mu$. In Appendix A, we show the logarithm term can be converted into a $\frac{v_0}{\Delta E^2}$ correction to the response rate. For the present analysis we will restrict our fields to be massless and conformally coupled which causes the potential v_0 to vanish identically and eliminate the logarithm term. As such we drop the term from our current analysis. Finally, utilizing Eqn. (2) and expanding out to s^4 in the denominator, we obtain the covariant and quasilocal expansion of the Hadamard form:

$$G^+[x(\tau'), x(\tau)] = \frac{1}{8\pi^2} \left[\frac{1}{-\frac{1}{2}s^2 - \frac{1}{24}(A^2 - R_{\alpha\beta} u^\alpha u^\beta + 48\pi^2 \langle \phi^2 \rangle_{ren}) s^4 + \mathcal{O}(s^6)} \right]. \quad (6)$$

Now that we have developed the Hadamard form, the next step will be to use its singularity structure to compute the temperature registered by an Unruh-DeWitt detector.

III. GENERALIZED TEMPERATURE

The transition rate of an Unruh-DeWitt detector is characterized by the response function [8]. Formally this is written as the Fourier transform of the two point function, i.e.

$$\Gamma \propto \int ds e^{-i\Delta E s} G^\pm[x(\tau'), x(\tau)]. \quad (7)$$

For a uniformly accelerated observer in Minkowski vacuum, the detector response will be thermal [3]. For a proper acceleration a , the Wightman function which characterizes this uniformly accelerated thermal state is well-known:

$$G_a^+[x(\tau'), x(\tau)] = \frac{-a^2}{(4\pi)^2} \frac{1}{\sinh^2(as/2)}. \quad (8)$$

A Wightman function of this form will produce a thermal response. Here, we note the Taylor expansion of this Wightman function will yield the same functional dependence as our Hadamard form. It should be noted that a uniform acceleration in one dimension is a necessary condition for this expansion to yield a thermal response at constant temperature, i.e $\partial_s T = 0$ [12]. Given the Taylor expansion of the thermal Wightman function

$$G_a^+[x(\tau'), x(\tau)] = \frac{1}{8\pi^2} \left[\frac{1}{-\frac{1}{2}s^2 - \frac{1}{24}a^2s^4 + \mathcal{O}(s^6)} \right]. \quad (9)$$

Now, we note that the coefficient on the quartic term determines the temperature of the resultant thermal response function. More precisely, the temperature for a thermal state is given by:

$$T_U^2 = \left(\frac{a}{2\pi} \right)^2 = -\frac{3}{8\pi^4} \lim_{s \rightarrow 0} \frac{\partial^2}{\partial s^2} \left[\frac{1}{s^2 G_a^+[x(\tau'), x(\tau)]} \right], \quad (10)$$

which is known as the Unruh temperature, observed by a constantly accelerated observer in the Minkowski vacuum [3].

The limit $s = \tau' - \tau \rightarrow 0$ is significant, as we expect a detector with a large gap $\Delta E \gg a$, which is on for a short period of time $\Delta\tau \ll a^{-1}$, to be only sensitive to this limit. The latter condition can help define an instantaneous notion of temperature, even if acceleration is not constant.

With this insight, let's turn to the generic Hadamard Wightman function derived above (Eq. 6). We can now define a generalized notion of *instantaneous* temperature:

$$T_*^2(\tau) \equiv -\frac{3}{8\pi^4} \lim_{s \rightarrow 0} \frac{\partial^2}{\partial s^2} \left[\frac{1}{s^2 G^+[x(\tau'), x(\tau)]} \right], \quad (11)$$

for a local observer, moving on an arbitrary trajectory, in a generic spacetime, and interacting with a generic Hadamard state of the quantum field. As we saw above, this matches the standard thermodynamic temperature for a thermal state. Plugging Eq. (6) into this definition, we find:

$$T_*^2(\tau) = \frac{1}{(2\pi)^2} [A^2 - R_{\alpha\beta} u^\alpha u^\beta + 48\pi^2 \langle \phi^2 \rangle_{ren}]. \quad (12)$$

Note we find the generalized temperature is written as the Pythagorean sum of the accelerated temperature $T_A = \frac{A}{2\pi}$, the state temperature $T_\phi^2 = 12 \langle \phi^2 \rangle_{ren}$, and a curvature temperature $T_R = \frac{\sqrt{-R_{\alpha\beta} u^\alpha u^\beta}}{2\pi}$. Moreover, if we had used the elementary averaging procedure the Raychaudhuri scalar would be replaced by the Ricci scalar yielding

$$\langle T_*^2(\tau) \rangle_{\text{el. ave.}} = \frac{1}{(2\pi)^2} \left[A^2 + \frac{1}{4} R + 48\pi^2 \langle \phi^2 \rangle_{ren} \right]. \quad (13)$$

Here, we now have a curvature temperature based off the Ricci scalar $T_R = \frac{\sqrt{R}}{4\pi}$. One particular application of this generalized temperature is that it provides insight into the state dependence via the vacuum polarization. Our vacuum polarization is given by

$$\langle \phi^2 \rangle_{ren} = \frac{1}{12} [T_*^2 - T_A^2] + \frac{1}{48\pi^2} R_{\alpha\beta} u^\alpha u^\beta. \quad (14)$$

In Page's method [7] to compute the renormalized vacuum polarization, he considered a conformally coupled massless scalar in an ultra-static Einstein spacetime where we have $R_{\mu\nu} = \Lambda g_{\mu\nu}$. For these classes of spacetimes, using either the Ricci or Raychaudhuri scalar, we reproduce Page's result

$$\langle \phi^2 \rangle_{ren} = \frac{1}{12} [T_*^2 - T_A^2] - \frac{\Lambda}{48\pi^2}. \quad (15)$$

IV. WHEN IS TEMPERATURE REAL?

For arbitrary spacetimes, the sign of the curvature correction to the temperature in Eq. (12) is constrained by the energy conditions:

The *strong energy condition* requires $R_{\alpha\beta}u^\alpha u^\beta \geq 0$ for time-like u 's, implying that curvature corrections to T_*^2 are always negative:

$$T_*^2(\tau) = T_A^2 + \frac{1}{(2\pi)^2} [-R_{\alpha\beta}u^\alpha u^\beta + 48\pi^2 \langle \phi^2 \rangle_{ren}] \leq T_A^2 + 12 \langle \phi^2 \rangle_{ren}. \quad (16)$$

Furthermore, the weaker *null energy condition* $R_{\alpha\beta}k^\alpha k^\beta \geq 0$, for all null vectors k^α implies that, even if the curvature correction is positive for an observer (in a spacetime violating the strong energy condition), it will become (arbitrarily) negative for an observer that moves fast enough (unless the null energy condition is saturated).

To see this explicitly, let us use Einstein equations for a perfect fluid with density ρ and pressure p . Then, Eq. (12) takes a remarkably simple form:

$$\begin{aligned} T_*^2(\tau) &= T_A^2 + \frac{1}{(2\pi)^2} \{48\pi^2 \langle \phi^2 \rangle_{ren} - 4\pi G_N [2(\rho + p)\gamma^2 - \rho + p]\} \\ &\leq T_A^2 + \frac{1}{(2\pi)^2} \{48\pi^2 \langle \phi^2 \rangle_{ren} - 4\pi G_N \gamma^2(\rho + 3p)\} \end{aligned} \quad (17)$$

where G_N is the Newton's constant of gravitation, and γ is the observer's Lorentz factor in the rest-frame of the fluid.

Indeed, for fast observers (large γ), we get $T_*^2 < 0$ assuming the null energy condition $\rho + p > 0$. There are two ways to interpret this result:

1. One possibility is that a conformally coupled massless field (assumed here) is not expected to directly couple to the Unruh-DeWitt detector, as such an interaction violates conformal symmetry. The detector could instead couple to e.g., field derivatives. We shall defer study of this possibility to future work.
2. One may also argue that such fast-moving detectors simply do not register a thermal response function, because they are not interacting with a thermal state.

Let us entertain the second possibility, and focus our attention on thermal gravitationally bound states. We can use Raychaudhuri equation (e.g., [15]) to substitute for $R_{\alpha\beta}u^\alpha u^\beta$:

$$R_{\alpha\beta}u^\alpha u^\beta = u^\mu \nabla_\mu (\nabla \cdot u) + 2(\Sigma^2 - \Omega^2) + \frac{1}{3}(\nabla \cdot u)^2 - \nabla_\mu a^\mu, \quad (18)$$

where we use the standard definitions for shear and vorticity tensors, Σ and Ω , as components of $\nabla_\mu u_\nu$ tensor. A thermal state does not have explicit time dependence, and thus must be in steady state. This implies that:

$$\Sigma^2 = \nabla \cdot u = 0, \quad \text{and} \quad T_{\text{global}} = T_*(x^\mu)/u^0(x^\mu) = \text{const.}, \quad (19)$$

where we assume that u^μ 's are the 4-velocities of the observers that see a steady-state thermal state (see e.g., [16]). Now, combining Eq's, (12) and (18-19), we find:

$$T_{\text{global}}^2 = \frac{1}{(2\pi u^0)^2} [48\pi^2 \langle \phi^2 \rangle_{ren} + a^\mu a_\mu + \nabla_\mu a^\mu + 2\Omega^2] = \text{positive constant}. \quad (20)$$

This equation provides a novel constraint on the renormalized field polarization in spacetimes with global thermal equilibrium, in terms of the velocity field of its thermal observers:

$$\langle \phi^2 \rangle_{ren} = \frac{1}{12}(u^0 T_{\text{global}})^2 - \frac{1}{48\pi^2} (a^\mu a_\mu + \nabla_\mu a^\mu + 2\Omega^2). \quad (21)$$

V. COSMOLOGICAL BACKGROUNDS

In a cosmological Friedman-Robertson-Walker (FRW) background with scale factor $a(\tau)$, Eq. (17) takes an interesting form for comoving observers:

$$T_*^2(\tau)_{\text{FRW}} = 12 \langle \phi^2 \rangle_{ren} + \frac{3}{(2\pi)^2} \frac{\ddot{a}(\tau)}{a(\tau)}, \quad (22)$$

which combines cosmic accelerations and the state temperature. The 4-acceleration for a comoving observer is zero. Moreover, if we recall that for a massless and conformally coupled scalar field, the vacuum polarization is given by $\langle \phi^2 \rangle_{ren} = -\frac{R}{288\pi^2}$ [17]. The cosmic acceleration can be written in terms of the Hubble constant via $\frac{\ddot{a}(\tau)}{a(\tau)} = \dot{H} + H^2$. Recalling that the Ricci scalar in an FRW spacetime is given by $12H^2 + 6\dot{H}$, we find the instantaneous temperature to be,

$$T_*^2(\tau)_{\text{FRW}} = \frac{1}{(2\pi)^2} [H^2 + 2\dot{H}]. \quad (23)$$

Considering the general expression for the temperature Eqn. (12), we can extend this result to include accelerated observers. Recalling the components of the Ricci tensor are given by $R_{00} = 3H^2 + 3\dot{H}$ and $R_{ii} = -a(t)^2(3H^2 + \dot{H})$ and the normalization of our 4-velocity yields $u_0^2 - a(t)^2 v^2 = 1$, we find the instantaneous temperature measured by an accelerated observer in agreement with [12]. Hence,

$$T_*^2(\tau)_{\text{FRW}} = \frac{1}{(2\pi)^2} [A^2 + H^2 + 2\dot{H}\gamma]. \quad (24)$$

VI. CONCLUSIONS

In this paper, we computed the temperature registered by an Unruh-DeWitt detector coupled to a Hadamard renormalizable massless quantum field. Employing a covariant and quasilocal expansion of the Hadamard form we find a temperature comprised of acceleration, curvature, and quantum state dependent terms. By restricting our geometry to static Einstein spacetimes we can reproduce the Page approximation. Moreover, we found a novel constraint on the renormalized field polarization in spacetimes with global thermal equilibrium. For FRW cosmologies, we reproduce the earlier result for the temperature determined by Obadia.

Appendix A: Logarithm term

The transition rate for an Unruh-DeWitt detector coupled to a Hadamard renormalizable quantum field is given, via Eqn. (7), by

$$\Gamma \propto \int ds e^{-i\Delta E s} \frac{1}{8\pi^2} \left[\frac{1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta + v_0 \sigma \ln(\tilde{\sigma}) + w_0 \sigma}{\sigma} \right]. \quad (A1)$$

The logarithmic term can be handled via integration by parts twice [14] yielding

$$\int ds e^{-i\Delta E s} v_0 \ln(\tilde{\sigma}) \approx \frac{v_0}{\Delta E^2} \int ds e^{-i\Delta E s} \frac{1}{\tilde{\sigma}}. \quad (A2)$$

Note that we disregarded the boundary terms produced via the integration by parts. These boundary terms can be formally shown to vanish via the use of a switching function to turn on/off the interaction. We also assumed the Hadamard coefficient is constant, i.e. $\partial_s v_0 = 0$. The last line was obtained under the assumption that Synge's world function takes the form $2\sigma \approx (\partial_s \sigma)^2$ in the quasilocal limit. Moreover we note that this term can be tuned away by increasing the energy gap such that $\Delta E \gg v_0$. Thus, under the above assumptions, the logarithm term doesn't contribute to the response function. Hence

$$\Gamma \propto \int ds e^{-i\Delta E s} \frac{1}{8\pi^2} \left[\frac{1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta + w_0 \sigma}{\sigma} \right]. \quad (A3)$$

Moreover, for conformally coupled massless scalar fields, the potential v_0 vanishes identically. Additionally, the definition for temperature used in this manuscript Eqn. (11) may not apply for a detector coupled to massive fields.

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- [1] L. Parker, The Creation of Particles by the Expanding Universe, Ph.D. thesis, Harvard University, 1966; *Nature* (London) **261**, 20 (1976) ;
 - [2] S. W. Hawking, *Nature* (London) **248**, 30 (1974); *Commun. Math. Phys.* **43**, 199 (1975).
 - [3] W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
 - [4] Marek J. Radzikowski, *Commun. Math. Phys.* **179**, 529 (1996).
 - [5] Yves Decanini and Antoine Folacci, *Phys. Rev. D* **78**, 04425 (2008).
 - [6] Chad R. Galley and B. L. Hu, *Phys. Rev. D* **72**, 084023 (2005).
 - [7] Don N. Page, *Phys. Rev. D* **25**, 1499 (1982).
 - [8] N. D. Birrell and P. C. W. Davies, *Quantum Field Theory in Curved Space* (Cambridge University Press, Cambridge, 1982).
 - [9] E. Poisson, A. Pound and I. Vega, *Living Rev. Rel.* **14**, 7 (2011) [arXiv:1102.0529 [gr-qc]].
 - [10] Stephen L. Adler, Judy Lieberman, and Yee Jack Ng, *Ann. Phys.* **106**, 279, (1977).
 - [11] Denis Bernard and Antoine Folacci, *Phys. Rev. D* **34**, 2286 (1986).
 - [12] Nathaniel Obadia, *Phys. Rev. D* **78**, 083532 (2008).
 - [13] Robert M. Wald, *Phys. Rev. D* **17**, 1477 (1977).
 - [14] Shin Takagi, *Prog. Theor. Phys. Supp.* textbf88, 1 (1986).
 - [15] N. Dadhich, gr-qc/0511123.
 - [16] C. Rovelli and M. Smerlak, *Class. Quant. Grav.* **28**, 075007 (2011) [arXiv:1005.2985 [gr-qc]].
 - [17] Mario A. Castagnino and Diego D. Harari, *Ann. Phys.* **152**, 85, (1984).